Solving Strong and Weak 4-in-a-Row

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***Abstract*—In this paper we first summarize knowledge about the standard (strong) version of the game 4-in-a-Row. It was previously shown that 4-in-a-Row is a draw on $4 \times n$ boards for $n =$ 4–8, and on the 5×5 board as well. Further we know that the game is a (first-player) win on the 5×6 board. Finally it is stated that 4-in-a-Row is a win on a 4×30 board. It is not known if and where there is a transition from drawn games to won games on $4 \times n$ boards for n ranging from 9–29. Using our k-in-a-Row solver we then show that the $4 \times n$ boards for $n =$ 9, 10, and 11 are wins. We provide the principal variations by our solver program for the 5×6 and 4×9 boards.**

Further we introduce a second version of the game, weak 4-in-a-Row (also called Maker-Breaker (MB) 4-in-a-Row), where the second player wins if he is able to prevent the first player from winning, but does not win by obtaining a 4-in-a-Row sequence himself. This game benefits the first player, since he can safely ignore any "threats" by the second player. Our results show that for weak 4-in-a-Row the first player already wins on the 5×5 and 4×7 boards. We also provide the principal variations by our solver program for these boards.

We then focus on the monotonicity property of winning positional games. It is widely believed that if a k-in-a-Row game is a win on some board, it will be a win on any larger board as well. This is denoted as a *monotone win*. As a consequence, if a k-in-a-Row game is a draw on some board, it will be a draw on any smaller board. For weak positional games the monotonicity property holds by definition. However, for the strong version of k-in-a-Row, the monotonicity property has never been formally proven. It is surely not true on arbitrary graphs, where enlarging the winning set might change a winning game into a draw. This phenomenon is known as the Extra Set Paradox. Still it is commonly believed that the monotonicity of winning does hold for rectangular boards. However, we show that even this is not true, at least not for arbitrary positions on a rectangular board. We demonstrate this with a counterexample.

To deal with the problem of non-monotonicity we propose an algorithmic solution. Suppose that k-in-a-Row is a win on some $m \times n$ board (assuming arbitrarily that $m \leq n$). To prove that this is a monotone win, we enlarge the board with a rim around the $m \times n$ board of width at most $k - 1$. On this board the first player is only allowed to play on the inner board, whereas the second player may use the whole board. We show that if with these constraints the first player still wins, the game is also a win on any larger board. Using this method we have shown that the 5×6 board indeed is a monotone win. For the 4×9 board we had to make a small adaptation, namely that the first player is allowed to use the rim, but only to respond to useless direct threats by the opponent. Moreover, any such stone of the first player in the rim may not contribute to winning variations in any way. Using this we prove that also the win on the 4×9 board is monotone. With these results strong and weak 4-in-a-Row are completely solved.

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I. INTRODUCTION

4-in-a-Row is a well-known game where two players called Black and White try to be the first to get a straight line of 4 consecutive stones of their color on some specified rectangular board. The winning lines can be horizontal, vertical or diagonal. By convention Black starts. 4-in-a-Row is an instance of a more general class of strong positional games. In the remainder of this paper, whenever we mention 4-in-a-Row, this strong version of the game is meant, unless explicitly stated otherwise. We first define our framework and then state some general properties of this class of games.

A. Background Theory for Strong mnk-Games

We start with providing some useful notions.

Definition 1: (Taken from [2], [3]) A *positional game* is a hypergraph (X, H), where the set X contains nodes forming the game board and $H \subseteq 2^X$ is a family of target subsets of X. During the game, two players alternately select one previously unclaimed element of the board. When the goal of the game for both players is to be the first to claim all elements of a target subset, the game is a *strong positional game*.

Standard k-in-a-Row games are examples of strong positional games. The target subsets are all the possible winning lines, also called *groups*, of k squares in a straight line. The first player who claims all elements of a group wins the game. If no player achieves a win, the game is a draw.

Definition 2: An mnk-game is a k-in-a-Row game played on an empty rectangular board with m rows and n columns.

Definition 3: An mnk-position is a board position that may occur in the course of an mnk-game.

A *position* therefore is a legal board position, mostly containing white and black stones, whereas a *game* is associated with the (empty) starting board.

Definition 4: A black (white) group is a consecutive horizontal, vertical, or diagonal line of k squares in an mnk-position that constitutes a possible winning line for Black (White), i.e., in which all squares are empty or black (white).

B. Some Theorems for Strong mnk-Games

Despite the simplicity of their rules, the theory on k-in-a-Row games is rather scarce. A useful theorem for strong positional games, due to John Nash, but probably first published in [7], is known as the *strategy-stealing argument*, which results in classifying mnk-games in two categories according to their outcomes.

Theorem 1: Any mnk-game is either a first-player win or a draw.

Proof: Suppose that some mnk-game would be a second-player win. So we assume that the second player has some strategy that assures him a win. But then the first player starts by playing an arbitrary first move, and subsequently uses the second-player's winning strategy. Whenever this strategy requires playing a square already filled by the first player, he just plays an arbitrary other possible move. Since placing an additional stone on the board never is a disadvantage for the player, this shows that the first player can win, which contradicts the assumption that there is a second-player winning strategy. Hence, there cannot exist a second-player winning strategy. ■

Using the convention that Black starts it means that an mnk-game either is a win for Black or a draw. We therefore further classify strong mnk-games just shortly as wins or draws. Note that this is not holding for arbitrary mnk-positions, which can be draws, or wins by either player. We will classify such positions therefore explicitly as black wins, white wins, or draws.

To prove that a set of groups is at most a draw for the first player, the well-known HJ-pairing strategy [7] can be used.

Definition 5: A *Hales-Jewett (HJ)-pairing* for a set of groups is an assignment of disjoint pairs of empty nodes (*markers*) to all groups, such that every group is covered by a marker pair. The second player hereby guarantees for every marker of a pair played by the first player to respond immediately by playing the second marker of the pair, thus covering every group of the set by at least one second-player stone, preventing a possible win by the first player in that set of groups.

Another useful theorem, as far as we know nowhere stated explicitly, is the following, and can easily be proven using the HJ-pairing strategy.

Theorem 2: Any mnk-game for $k \geq 3$ with $m < k$ and/or $n < k$ is a draw.

Proof: If both $m, n < k$, then the game is trivially drawn, since there are no winning sets. Otherwise, suppose arbitrarily that $m < k$ and $n \geq k$. Then the only possibility for getting k-in-a-Row is as a horizontal group with k connected black stones (White can never win, according to Theorem 1). The following strategy by White, however, prevents Black from getting three or more connected stones in a (horizontal) row. For every row of the board, apply the HJ-pairing strategy to every pair of neighbouring squares, starting at the left side. If the width is odd, leave the last square in a row unmarked. After any black move on a marked square White responds by playing the other square of the pair. If the black move is on an unmarked square, White plays an arbitrary possible move. Using this strategy White prohibits Black from occupying more than 2 connected squares in a row, thus preventing a black win. In case $m \geq k$ and $n < k$ a similar strategy is used in columns instead of rows. ■

We next state two conjectures for strong mnk-games:

Conjecture 1: If some strong mnk-game is a draw, then any strong $m'n'k$-game with $m' \leq m$ and $n' \leq n$ is a draw.

Conjecture 2: If some strong mnk-game is a win, then any strong $m'n'k$-game with $m' \geq m$ and $n' \geq n$ is a win.

These two conjectures together denote the *monotonicity property* of winning strong k-in-a-Row games (see Sections IV and V for more on this). Although this property seems very plausible (and in fact in many papers on mnk-games is used), it has so far not been proven formally. However, it can be proven for a related class of positional games, namely *weak* positional games.

C. Background Theory for Weak mnk-Games

In many publications on positional games also the category of *weak* positional games is defined [2], [3], [8]. They differ from strong positional games only in their winning conditions.

Definition 6: A *weak* positional game is a positional game with the following ending conditions: the goal of the first player is similar as in strong positional games, namely claiming all squares of a winning set, however **not necessarily first**. The goal of the second player is to stop the first player from reaching his goal; if the second player succeeds, we call it a second-player win.

So, weak positional games and positions have just two possible outcomes, a first-player win (quite similar as in a strong game) and a second-player win (resembling a draw in the strong version of the game). We therefore further classify weak mnk-games just shortly as wins or losses (both from the viewpoint of the first player).

By the nature of the different goals of both players weak positional games are also often denoted as *Maker-Breaker* (MB) games, where the goal of Maker (Black) is to claim a complete winning set, whereas the goal of Breaker (White) is to prevent Black from winning. Then the strong positional games can analogously be denoted as *Maker-Maker* (MM) games.

Based on their definitions there is an obvious relationship between the outcomes of strong and weak mnk-games.

Theorem 3: If some strong mnk-game is a first-player win, then the weak mnk-game is a first-player win as well.

Proof: The proof of this theorem is trivial. Since Black has a winning strategy in the strong mnk-game, he just can use the same strategy in the weak version of the game. Since White can not prevent a black win in the strong version, he also can not prevent it in the weak version, because his abilities in the weak game are just a subset of those in the strong game (in particular, there are no white threats that Black must answer). ■

In many cases, Black wins faster in a weak game than in its strong version, but he always could ignore a faster win and use the same strategy. Note that the opposite of Theorem 3 is not valid: if a weak mnk-game is a first-player win, then the strong version not necessarily is a win also, since White might be able to prevent that using threats not available in the weak version.

Theorem 4: If some weak mnk-game is a second-player win, then the strong mnk-game is a draw.

Proof: The proof of this theorem is also trivial. Since White has a strategy to block any black possible win in the weak mnk-game, he just can use the same strategy in the strong version of the game. This means White has a strategy preventing the first player from winning without needing any white threats. ∎

Again, in many cases White has a faster way to prevent Black from winning a strong game than its weak version, but he needs not to. Also note that the opposite of Theorem 4 is not valid: if a strong mnk-game is a draw, then the first player might still win the weak version of the game, since he may safely ignore any white threats.

D. Some Theorems for Weak mnk-Games

Conjectures 1 and 2, though not being proven for strong positional games, can be proven quite easily for weak positional games:

Theorem 5: If some weak mnk-game is a second-player win, then any weak $m'n'k$-game with $m' \leq m$ and $n' \leq n$ is a second-player win.

Proof: The proof of this theorem is trivial. Since White has a blocking strategy on the $m \times n$ board, he just can use the same strategy on the $m' \times n'$ board. If the strategy demands playing outside the $m' \times n'$ board, White just plays an arbitrary move. When a demanded white move has already been played before, he also plays an arbitrary other move. Since additional white stones never can hurt White, this proves that Black never can reach a win. ∎

Theorem 6: If some weak mnk-game is a first-player win, then any weak $m'n'k$-game with $m' \geq m$ and $n' \geq n$ is a first-player win.

Although this is a quite obvious result, the proof seems not quite trivial and was still lacking to our knowledge. Using the previous theorem it is surprisingly simple though.

Proof: Suppose the $m'n'k$-game is a second-player win. This means that the second player has a blocking strategy and according to Theorem 5 any equal or smaller game then is a second-player win. But the mnk-game is a first-player win, falsifying the assumption. Consequently, the $m'n'k$-game is a first-player win also. ∎

II. Results for Strong 4-in-a-Row

Results for strong 4-in-a-Row in the literature are scarce. In [4] it was shown by a strategy found by C.Y. Lee that the 554-game is a draw. It further was stated that the 4×30 game would be a win.[1]

A systematic computer study [11] with a predecessor of our current 4-in-a-Row solver showed that 4-in-a-Row on $4 \times n$ boards for $n \leq 8$ are draws. The 4×9 board could not be solved then. Further it was shown that the 5×5 board is a draw, while on the 5×6 board the game is a first-player win. Therefore it is known that the 5×6 board is the smallest board on which the first player wins.

[1]A reference to a thesis [10] was given, but so far we were not able to track down the source for inspection of this curious result.

Our new solver has much more elaborate (deeper) rules and other domain knowledge, and a more sophisticated Hales-Jewett pairing strategy. This results in proving outcomes often earlier than with our previous solver. For details on the knowledge used in our current solver we refer to [13].

Using our current k-in-a-Row solver we solved 4-in-a-Row on all $4 \times n$ boards for $4 \leq n \leq 11$ and the 5×5 and 5×6 boards. Results are in Table I.[2]

TABLE I
Number of Visited Nodes Solving Strong 4-in-a-Row (Maker-Maker version)

Board size	outcome	# of nodes
4×4	draw	1
4×5	draw	189
4×6	draw	4,452
4×7	draw	69,717
4×8	draw	1,498,074
4×9	win	4,614,766
4×10	win	60,178,371
4×11	win	536,741,564
5×5	draw	23,076
5×6	win	61

A. The 5×6 and 4×9 Boards are First-Player Wins

From the results in Table I we see that the 5×6 board is an easy win. Only 61 nodes are needed to prove the win within a few milliseconds. In the following analyses we use Chess-like notation, where squares are indicated by their column, numbered from left to right as a, b, c, etc, and their row, numbered from bottom to top by 1, 2, 3, etc. The main variation of our program is given in Figure 1.

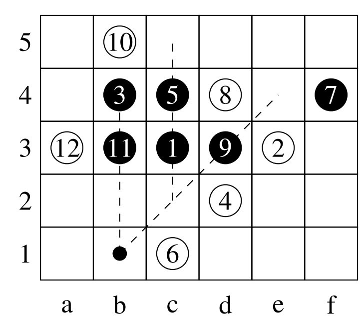

Fig. 1. Principal variation on the 5×6 board.

1. c3 2. e3! strongest defense **3. b4** threatening to make a winning Open-3[3] **4. d2** best defense; the only alternatives 4. e1 and 4. a5 lose faster **5. c4!** threatening to make two winning Open-3s simultaneously, at c2 and d4 **6. c1** strongest resistance, by parrying one Open-3 threat and making a direct white threat (the only other escape 6. f4 loses faster to 7. c1

[2]To gain more evidence that all larger boards including a 5×6 subboard are indeed also wins, we also solved all $m \times n$ boards with $5 \leq m \leq 10$ and $6 \leq n \leq 10$. They all are wins, needing between 31 and 215 nodes to prove the win.

[3]An Open-3 is a straight sequence of 3 stones of the player with an empty square at both sides, so it is a double threat of which only one can be parried, thus resulting in a win.

(forced) 8. c2 (forced) whereafter 9. d4 achieves a winning Open-3) **7. f4** forced **8. d4** parrying the other Open-3 threat (the other two ways to parry the Open-3 threat by playing 8. a4 or 8. e4 lose faster) **9. d3** with again an Open-3 threat, this time at e2 **10. b5** strongest defense; other defenses like 10. e2 or 10. f1 immediately lose to Black's double threat at b3; alternatively White can first make an intermediate direct threat by 10. c5, but this loses after 11. f2 (forced) 12. f3 (best way to prevent the Open-3) 13. b3 14. a3 (forced) 15. b5 with two direct threats; the other intermediate direct threat by 10. f2 loses faster after 11. c5 (forced) 12. c2 (forced) 13. e2 (forced, but at the same time an Open-3 win) **11. b3 12. a3** forced, yielding the final position in Figure 1. In this position the program establishes a win-in-5, indicated with the dashed lines (**13. b1** direct threat **14. b2** forced **15. c2** wins by two direct threats at c5 and e4).

This confirms (as already known from literature [9], [14]) that the 5×6 board is the smallest board on which Black can win.

Also, the 4×9 board is a win. For this proof some 5 million nodes in around 58 seconds were investigated. The main variation of our program is given in Figure 2.

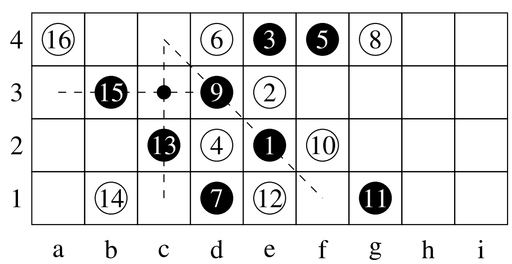

Fig. 2. Principal variation on the 4×9 board.

1. e2 2. e3! strongest resistance **3. e4! 4. d2 5. f4** threatening an Open-3 and at the same time preventing a white direct threat **6. d4** strongest defense **7. d1** threatening a double direct threat at g4 **8. g4** best; a strong alternative is 8. f3, after which 9. d3 eventually wins like in the main variation **9. d3!** very strong move, threatening to win by a direct threat at b1 followed by a winning Open-3 at c1, but simultaneously a direct threat at f1 followed by a winning Open-3 at e1 **10. f2** strongest way to delay the loss by using a direct threat **11. g1** forced **12. e1** the other two options 12. f1 and 12. c4 immediately lose to 13. b1 14. c2 15. c1 **13. c2** but now Black wins by a series of direct forcing moves **14. b1** forced **15. b3 16. a4** forced, yielding the final position in Figure 2. In this position the program establishes a win-in-5, indicated with the dashed lines (**17. c3** direct threat **18. a3** forced **19. c4** winning by a double direct threat (Black wins at c1 or f1).

B. Correctness of the Results

There are several methods we have used to check the correctness of our results.

First, all results are in agreement with previous results from literature [4], [9], [11], [14]. Moreover, shortly after the results mentioned in this paper became available, a student at our department independently built a solver with less knowledge, but including a threat-sequence searcher, confirming our results [5].[4]

Second, for many boards solved we also solved them using less rules (disabling the more complex rules), resulting in investigating more nodes, but always leading to the same results.

Third, and most importantly, for all rules and pairing strategies in our program we incorporated a test feature that checks all conditions of the rule or strategy in question and notes the result, but then continues the search without using the result and comparing the values with and without applying the rule or strategy. This feature is very useful in debugging and guaranteeing the correctness of each rule and strategy separately. In the final version of the program no discrepancies were encountered.

III. Results for Weak 4-in-a-Row

We also adapted our solver to be able to solve weak mnk-games. Most importantly it meant a change in knowledge rules, notably regarding the conditions for applicability with respect to the absence of white threats.

Using our new k-in-a-Row solver we solved weak 4-in-a-Row on $4 \times n$ boards for $4 \leq n \leq 7$ and the 5×5 board. Results are in Table II.

TABLE II
Number of Visited Nodes Solving Weak 4-in-a-Row (Maker-Breaker version)

Board size	outcome	# of nodes
4×4	loss	1
4×5	loss	123
4×6	loss	2,978
4×7	win	55,635
5×5	win	7

We observe that weak 4-in-a-Row is already a black win on the 4×7 board (the smallest $4 \times n$ board) and on the 5×5 board (the smallest board at all).

A. The 5×5 and 4×7 Boards are First-Player Wins

The 5×5 board is a trivial win. Only 7 nodes are needed to prove the win within a few milliseconds (see the position in Figure 3).

The main variation is **1. c3**, after which every white response leads to a position immediately recognized as a win. For instance, after the response **2. b4**, Black has a win-in-7, e.g., by playing **3. d3**, Open-3 threat, **4. b3 5. d4** a double Open-3 threat, **6. b2 7. d2** Open-3, and Black wins next move at d1 or d5. Note that Black need not respond to White's three consecutive stones in the b-file, since a White sequence of 4 stones does not win for White.

Since a win for weak 4-in-a-Row is monotone, it means that all boards containing a 5×5 subboard, i.e., all $m \times n$ boards with $m, n \geq 5$, are wins.

[4]In his solver, the 4×9 win was found investigating 79 million nodes in 358 seconds.

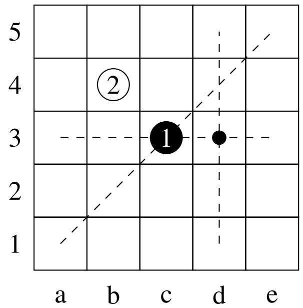

Fig. 3. Principal variation on the 5×5 board.

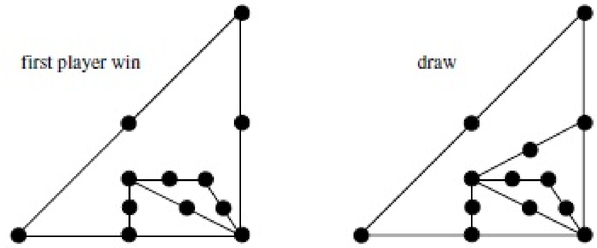

Fig. 5. Illustration of the Extra Set Paradox (taken from [3]).

Regarding $4 \times n$ boards we already find that the 4×7 board is a win. For this proof just some 56 thousand nodes were investigated in around 1 second. The main variation is given in Figure 4.

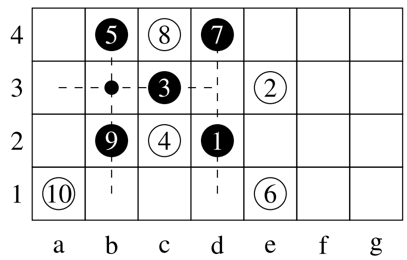

Fig. 4. Principal variation on the 4×7 board.

1. d2 2. e3 3. c3 4. c2 5. b4 direct threat **6. e1** forced **7. d4** Open-3 threat **8. c4** best reply **9. b2** direct threat **10. a1** forced, yielding the final position in Figure 4. In this position the program establishes a win-in-5, indicated with the dashed lines, starting with **11. b3** direct threat **12. b1** forced **13. d3** double direct threat, winning at a3 or d1.

IV. The Extra Set Paradox

So far it has not been proven that a win for strong mnk-games is monotone, since it might be the case that on some board White effectively can use a threat, which he can not use on a smaller board. This is an example of Beck's *Extra Set Paradox* [3]. In Figure 5 we repeat his example, which is a small adaptation of an example by Gardner [6] as originally notified to the latter by Austin and Knight. Note that in this example all straight lines with three nodes are winning sets (so this is a 3-uniform example of the Extra Set Paradox).

In the left graph the first player (say Black) wins by taking the right-down corner. If the second player (say White) does not respond in the large triangle, Black continues by taking the upper corner in the large triangle and easily wins. If White does respond in the large triangle, Black wins likewise in the upper inner small triangle. In the right graph however, taking the right-down corner by Black is responded by White by taking the inner node with degree 4. If Black now continues by taking the upper corner in the large triangle, White's forced response poses a direct threat to which Black has to defend. It can easily be verified that this and other optimal variations lead to a draw. This example shows that adding winning lines to a position can transform the win into a draw. Gardner [6] then remarks that his original statement that "if the first player always wins on a board of a certain size, he also wins on any larger board" therefore does not hold on arbitrary graphs, but still holds on square boards. However, this last statement is not formally proved so far and it surely does not hold for arbitrary positions on square boards. Consider the 554-position in Figure 6 (left).

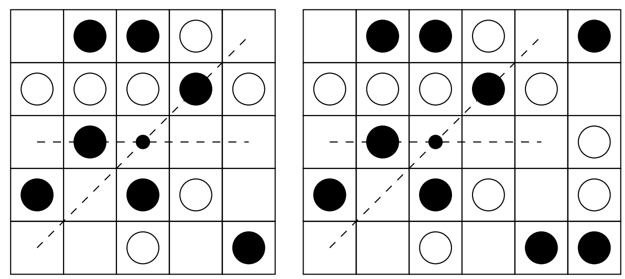

Fig. 6. Two example positions: a win for Black on the 5×5 board (left), but a draw on the 5×6 board (right).

In this position Black wins by playing c3, indicated with the small dot. This move makes two Open-3 threats, indicated by the dashed lines. One Open-3 threat can be parried, but then Black wins by an Open-3 in the other line. However, when we extend this position with an additional column to the right (the position in Figure 6 (right)), then White can respond with e3, which is a direct threat, to which Black must respond at f4, after which White's b2 draws.

Though not formally proved we still strongly believe that Conjectures 1 and 2 hold.[5] Note that in weak k-in-a-Row games (in fact all weak games) the Extra Set Paradox is not applicable, since the second player has no threats to which the first player must respond to avoid losing. As a consequence, wins in weak games are always monotone.

[5]Two reasons why we feel these conjectures most likely hold are based on evidence: 1) on the larger board Black, who has the initiative, has even more room for maneuvring and avoiding threats by White; and 2) the conjectures are completely in accordance with all experimental evidence so far, i.e., up till now there is not a single strong mnk-game known to be a draw, for which a smaller board is a win.

V. An Algorithmic Solution

To solve the problem of possible non-monotonicity of winning in strong k-in-a-Row games we elaborated on the following idea. The only reason that a strong win on some $m \times n$ board can be non-monotone is that White on larger boards might have winning sets (and so possible threats) that are not present at the $m \times n$ board (the Extra Set Paradox), because otherwise Black could use exactly the same winning strategy on the larger boards as well. As a consequence, when Black wins on an $m \times n$ board even when White is allowed to play "outside the board" but Black not, it follows that Black's win is monotone.

We therefore define a new set of games.

Definition 7: A strong mnk-rs-game is a strong k-in-a-Row game played on an empty $(m+2r) \times (n+2s)$ board, where the first player is only allowed to play on the "central" $m \times n$ board and the second player is allowed to use the full board, which consists of the central board with a two-sided horizontal rim of width r and a two-sided vertical rim of width s.

This means that if White has a timely direct-threat involving a square on a rim, Black cannot defend and loses such a position. As a small test-by-hand: for the strong 343-game (the smallest board size where the first player wins the TickTackToe game) it is easily shown that Black still has a win for any rim size, since White has no time to play in the rim at all (if Black starts in one of the centre squares, White can not play in the rim, since Black then obtains a winning Open-2; after that all black moves are direct threats leading to a win). Hence, Black wins any $mn3$-game on boards 3×4 and larger, and consequently the win of the strong 343-game is monotone.

We have built a rim-version of our solver of strong 4-in-a-Row, implementing the rules for the rim version of the game and of course adapting all knowledge rules used in such a way that wins by Black only involve groups completely within the central board, but that threats by White that are incorporated in the conditions of the rules may include squares in the rim.

The question now is how large such rims at least must be. It is evident that a rim of size k is over-sufficient, since this is only useful for White if he is allowed to obtain a sequence of k consecutive white stones in a straight line across a full rim of size k. But in order to reach that he should play at least k times in a rim, meaning that Black has the opportunity to play at least k additional moves before White can reach such a win, enough for Black to win. Therefore a rim of size at most $(k-1)$ is sufficient.

The following three theorems state this idea formally.

Theorem 7: If a strong mnk-game is a first-player win and the mnk-rs rim-game is not a first-player win (so it is a draw or a second-player win), then every mnk-$r's'$-game with $r' \geq r$ and $s' \geq s$ is not a first-player win.

Proof: Given that the second player can prevent the first player from winning the mnk-rs-game, he can use exactly the same strategy on any $m \times n$ board with a larger rim and prevent the first player from winning, by just not using the additional rim surplus. ∎

Theorem 8: If a strong mnk-rs-game is a win, then the strong $(m+2r)(n+2s)k$-game is a win as well.

Proof: This proof is trivial. The first-player can use in the $(m+2r)(n+2s)k$-game the same strategy as in the mnk-rs-game, thus guaranteeing him a win, which the second player obviously can not avoid, even not by playing outside the central $m \times n$ board. ∎

Theorem 9: If a strong mnk-game is a win and the strong mnk-$(k-1)(k-1)$ rim-game is also a win, then the win of the mnk-game is monotone.

Proof: Given that the first player wins the strong k-in-a-Row game on the $m \times n$ board and still wins the rim-game with rims of size $(k-1)$, it follows from Theorems 7 and 8 that all strong $m'n'k$-games with $m \leq m' \leq m + 2(k-1)$ and $n \leq n' \leq n + 2(k-1)$ are wins. Now suppose an even larger board is not a win. Then it must be the case that White has a threat involving a square outside the $(k-1)$ rim. But such a threat necessarily concerns a winning line completely in the rim, for which White has no time as shown above. So this contradicts the assumption that this larger game is not a win. Hence all larger boards are wins also, which means that the mnk-game is a monotone win. ∎

Due to Theorem 7 to investigate if some strong mnk-game win is monotone it is useful to first apply smaller rims before applying the $(k-1)$ rims. Therefore it is natural to start with rims of size 1 and as long as the rim game is still a first-player win, increment the rim up to $(k-1)$. Only if the latter game is still won by the first player, it is guaranteed that the mnk-game win is monotone.

It is not guaranteed that this idea works for wins in arbitrary strong mnk-games, so we have to investigate this experimentally for relevant games explicitly. The first experiment is to determine if the strong 564-game is a monotone win. In Table III we give the results of solving 564-rr games with increasing rim size r.

TABLE III
Number of Visited Nodes Solving Strong 4-in-a-Row on the 5×6 Board with Rims of Different Sizes

Rim size	outcome	# of nodes
0	win	61
1	win	7,084
2	win	6,393
3	win	18,833
4	win	18,833

We see that the strong 564-rr-games are all first-player wins for rim sizes up to 3. We may conclude that the win of the 564-game is monotone, hence that strong 4-in-a-Row is a win on any $m \times n$ board with $m, n \geq 5$ and $max(m, n) > 5$. We also note that the number of nodes investigated for rim size 3 and 4 are exactly equal. This supports our claim that a rim size of $(k-1)$, i.e., 3 for 4-in-a-Row, is large enough to investigate a potential monotone win.

We next applied this method to strong 4-in-a-Row games on $4 \times n$ boards. In this case it is sufficient to add only vertical rims, since we are only investigating the monotonicity of wins of boards with a fixed number of 4 rows. Results are in Table IV.[6]

TABLE IV
NUMBER OF VISITED NODES SOLVING STRONG 4-IN-A-ROW ON $4 \times n$ BOARDS WITH n =9, 10, AND 11, FOR RIMS OF DIFFERENT SIZES

Board	Rim size	outcome	# of nodes
4×9	0	win	6,191,234
4×9	1	no win	94,888,015
4×10	0	win	83,305,748
4×10	1	no win	13,102,532,623
4×11	0	win	836,261,749
4×11	1	unknown	$\gg$ 25,000,000,000

Unfortunately, for the 4×9 board (the smallest $4 \times n$ board where the first player has a win) we find that already with a rim of size 1 (and due to Theorem 7 therefore with any larger rim) the second player can prevent the first player from winning (see Table IV, upper part). This was not completely unexpected, since we know from our results that 4-in-a-Row on the 4×9 board involves a very complicated win for the first player, mainly since the strongest way to win, by posing Open-3 threats, is only possible in the horizontal direction.

We then did the same experiment for the 4×10 board, but again already with a rim of size 1 White can prevent Black from winning (Table IV, middle part). For 4×11 the number of investigated nodes for a rim size of 1 already becomes excessively larger than reasonably feasible (see Table IV, lower part), let alone (if this board would still be a win for Black) for rim sizes of 2 and 3.

Investigating the results so far we found as reason for this behavior the possibility for White to use the rim for *useless* threats, that Black cannot parry due to his handicap. We then made an adaptation to our rim-version of the program, where Black is allowed to defend a direct threat in the rim, but further is still not allowed to use the rim. This prevents (sequences of) useless threats by White, but cannot prevent White from any potential to win or to hamper a black win using the rim. Moreover, responses by Black in the rim may not be involved in the final black win in any won variations. We denote such adaptation of a rim game by rim+. The next theorem formally proves that Theorem 9 still holds for this modified rim+ version of k-in-a-Row rim games.

Theorem 10: If a strong mnk-game is a win and the strong mnk-$(k-1)(k-1)$ rim+ game is also a win, then the win of the mnk-game is monotone.

Proof: The proof is similar as the proof of Theorem 9 since (sequences of) threats by White in the rim are useless when Black still wins the rim games. ■

Using this version we repeated the previous experiments. The results are shown in Table V.

In the upper part of the table we find that for the 4×9 board it still holds that White can prevent Black from winning with a rim size of at least 1. In the lower part of the table, it is shown though that for the 4×10 board White is no longer able to prevent Black from winning using rims of sizes up to 3 or more. The results once more indicate that increasing the rim size from 3 to 4 for 4-in-a-Row hardly has any effect, again supporting our claim that a rim of size $k-1$ is sufficient. Using Theorem 10 the results show that the black win on the 4×10 board is monotone, and since Black also wins on the 4×9 board, it follows that the black win on the 4×9 board is monotone.

TABLE V
NUMBER OF VISITED NODES SOLVING STRONG 4-IN-A-ROW ON $4 \times n$ BOARDS WITH n =9 AND 10 USING THE RIM+ VERSION, FOR RIMS OF DIFFERENT SIZES

Board	Rim size	outcome	# of nodes
4×9	0	win	6,191,234
4×9	1	no win	100,355,209
4×10	0	win	83,305,748
4×10	1	win	312,543,420
4×10	2	win	386,574,878
4×10	3	win	689,771,099
4×10	4	win	689,607,829

VI. CONCLUSIONS AND FUTURE RESEARCH

In this paper we have closely investigated the simple (in rules) but yet complicated (in strategy) game of 4-in-a-Row. We have obtained four main conclusions, all four connected to a different version of the game.

Continuing earlier work on the strong version of the game using a mediocre knowledge-based program we now have used a more deep and refined knowledge-based solver. We were not only able to reproduce within a fraction of a second the fact that the 5×6 board is indeed the smallest board on which Black can win, but also showed with some more effort (about a minute) that the 4×9 board is the smallest $4 \times n$ board on which Black wins. For both boards we have given our program's principal variations, clearly exhibiting the inherent complexity of the game.

Next we showed that weak 4-in-a-Row is already a black win on the 5×5 (the smallest) and 4×7 (the smallest $4 \times n$) boards. These wins are easier, since White can not use threats by himself, but only can block Black by claiming at least 1 stone in every group. By definition these wins are monotone and therefore weak 4-in-a-Row is completely solved.

We also investigated the problem posed by the Extra Set Paradox, which means that wins in the strong version are not necessarily monotone (at least not proven so far, though they probably are, evidenced by all available results). This means that from solving the 5×6 and 4×9 boards we may not conclude that any larger board incorporating a 5×6 or 4×9 board is a black win also. For the 5×6 board we formulated an algorithmic solution based on a rim version of the game, where Black can only claim squares on a central board, whereas White can claim squares on the whole board, a central board with two-sided rims. The results showed that Black still wins the game, even if White is allowed to use a rim of size 3 or more. Therefore, the black win in the 5×6 game is monotone.

[6]The slightly larger numbers of investigated nodes for the boards with rim size 0, compared with Table I, are due to a small difference in move ordering in this version of the program.

For the $4 \times n$ boards with $n \geq 9$ this did not work so far, and we adapted our program so that Black is allowed to respond to direct threats in a rim as long as they are not useful for White (i.e., do not prevent black wins) and do not support Black in obtaining a win. With this version, Black still was unable to win the 4×9 board with rims of size 1 or larger. However, from 4×10 onwards Black wins with rims of sizes up to 4. Combined with the black win on the 4×9 board without a rim, this means that the black win on the 4×9 board also is monotone.

By the previous two results it now is known that Black wins on any $4 \times n$ board with $n \geq 9$ and on any $m \times n$ board with $m, n \geq 5$ and $max(m, n) > 5$, This means that also strong 4-in-a-Row is now completely solved.

Regarding future research, we first of all will elaborate on the current components of our solving program and enhance these where possible (for which we still have many ideas).

Further we want to implement two main ideas. The first one is to combine our solver which is strong in quickly finding won positions using knowledge rules [13] with a form of threat-space search as applied by Charatsidis for 4-in-a-Row [5] and by Allis et al. for Go-Moku (5-in-a-Row on a 15×15 board with slightly special rules) [1]. This combination is expected to find winning variations even much quicker. The second idea is to implement the strategy of Set Matching [12] which is an enhancement of the Hales-Jewett pairing strategy [7]. Such implementation is expected to find drawing variations much quicker.

We then want to use these advanced techniques to also investigate larger versions of strong k-in-a-Row games (for $k = 5, 6$ or 7) for which we do not have full knowledge yet, including the topic of monotonicity of wins. Of course it would be great if a general proof of the monotonicity property of k-in-a-Row on rectangular boards was found.

Finally we also aim at solving weak (Maker-Breaker) and maybe other interesting versions for such games.

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